THE INTRINSIC GEOMETRY OF ALMOST CONTACT METRIC MANIFOLDS

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Abstract. In this paper the notion of the intrinsic geometry of an almost contact metric manifold is introduced. Description of some classes of spaces with almost contact metric structures in terms of the intrinsic geometry is given. A new type of almost contact metric spaces, more precisely, Hermitian almost contact metric spaces, is introduced.

Key words: almost contact manifold, Sasakian manifold, K-contact manifold, the intrinsic geometry of almost contact metric manifolds.

Introduction

The research of the geometry of manifolds with almost contact metric structures has been begun in the fundamental papers by Chern [1], J. Gray [2] and Sasaki [3]. Almost contact metric structures constitute the odd-dimensional analog of almost Hermitian structures. There are a lot of important interplays between these structures. In the same time, the geometry of almost contact metric structures is appreciably different from the geometry of almost Hermitian structures and its study requires in principle new tools. The results obtained in this area before 1976 are reflected in full measure in the book [4]. Important contribution to the development of the geometry of almost contact metric spaces inserted V.F. Kirichenko and his students, see [5, 6]. One can get the impression of the last achievements in this area and about applications to theoretical physics from the works [7, 8].

In the present paper, the notion of the intrinsic geometry of a manifold with an almost contact metric structure is introduced. In the terminology developed by V.V.Wagner [10], the manifold with an almost contact metric structure is a nonholonomic manifold of codimension 1 with additional structures. These structures Wagner called intrinsic. The notion of the intrinsic geometry of a nonholonomic manifold was defined by Schouten as the properties that depend only on the parallel transport in the nonholonomic manifold and on the closing of the nonholonomic manifold in the ambient manifold.

In [9] Wagner writes: "Schouten shows the possibility of the direct construction of the intrinsic geometry of a nonholonomic manifold without usage of the parallel transport in the ambient space. In the case of metric nonholonomic manifold, the intrinsic geometry is defined by assigning a closing and a metric on local tangent spaces". Developing the intrinsic geometry of a nonholonomic manifold, Wagner defines and investigates the curvature tensor of a nonholonomic manifold that generalizes the Schouten curvature tensor. The curvature tensor, which later was called the Wagner curvature tensor, first was constructed for a nonholonomic metric manifold of arbitrary codimension [9], and then this result was specified for the case of a nonholonomic manifold of codimension 1 endowed with an intrinsic linear connection [10]. Wagner used his theory of the curvature of nonholonomic manifolds for solving some problems of classical mechanics and calculus of variations. In the present paper we propose to use the methods of nonholonomic geometry developed by Wagner for investigation the geometry of manifolds with almost contact metric structure. The new approach allows to pick out new types of spaces. For example, we give the definition of an Hermitian almost contact metric spaces.

The already known results obtain new description on the language of the intrinsic geometry. Following the ideology developed in the works of Schouten and Wagner, we define the intrinsic geometry of an almost contact metric space X as the aggregate of the properties that possess the following objects: a smooth distribution D defined by a contact form η; an admissible field of endomorphisms φ of D (which we call an admissible almost complex structure) satisfying $\varphi^2 = -1$; an admissible Riemannian metric field g that is related to φ by $g(\varphi\vec{X}, \varphi\vec{Y}) = g(\vec{X}, \vec{Y})$, where $\vec{X}$ and $\vec{Y}$ are admissible vector fields. To the objects of the intrinsic geometry of an almost contact metric space one should ascribe also the objects derived from the just mentioned: the 2-form $\omega = d\eta$; the vector field $\vec{\xi}$ (which is called the Reeb vector field) defining the closing $D^\perp$ of D, i.e. $\vec{\xi} \in D^\perp$, and given by the equalities $\eta(\vec{\xi}) = 1$, $\ker\omega = \mathrm{span}(\vec{\xi})$ in the case when the 2-form ω is of maximal rank; the intrinsic connection ∇ that defines the parallel transport of admissible vectors along admissible curves and is defined by the metric g; the connection ∇^1 that is a natural extension of the connection ∇ which accomplishes the parallel transport of admissible vectors along arbitrary curves of the manifold X.

The paper consists of two sections. In the first section we provide the basic concepts of the theory of manifolds with almost contact metric structure. We introduce the notion of the adapted coordinate system. The adapted coordinates play in the geometry of the nonholonomic manifolds the same role as the holonomic coordinates on a holonomic manifold, see e.g. [10]. The adapted coordinates are extensively used in the geometry of foliations [11]. Next we introduce the notion of an admissible (to the distribution D) tensor structure. An admissible tensor structure is an object of the intrinsic geometry of a nonholonomic manifold [10]. In the literature on the geometry of the fibering spaces, the admissible tensor structures are usually called semi basic. We give some information about the intrinsic connections compatible with admissible tensor structures. Among the connection compatible with the admissible Riemannian metric, we study the connections compatible with an admissible almost complex structure. We discuss the connection over a distribution that was introduced in [12, 13] and applied in [14, 15] to manifolds with an admissible Finsler metric.

In the second section we expound some of the main theses of the geometry of almost contact metric spaces in terms of the intrinsic geometry. It is shown that the almost contact metric structure defined in the intrinsic way corresponds to a certain almost contact metric structure defined in the usual way. The intrinsic connection is used for description and characterization of the normal and Sasakian structures.

1. Admissible tensor structures and intrinsic connection compatible with them

Let X be a smooth manifold of an odd dimension n. Denote by $\Xi(X)$ the $C^\infty(X)$-module of smooth vector fields on X and by d the exterior derivative. All manifolds, tensors and other geometric objects will be assumed to be smooth of the class C^∞. For simplification, in what follows we call tensor fields simply be tensors. An almost contact metric structure on X is an aggregate $(\varphi, \vec{\xi}, \eta, g)$ of the tensor fields on X, where φ is a tensor field of type $(1,1)$, which is called the structure endomorphism, $\vec{\xi}$ and η are vector and covector, which are called the structure vector and the contact form, respectively, and g is a (pseudo-)Riemannian metric. Moreover,

$$\eta(\vec{\xi}) = 1, \quad \varphi(\vec{\xi}) = 0, \quad \eta\circ\varphi = 0,$$
$$\varphi^2\vec{X} = -\vec{X} + \eta(\vec{X})\vec{\xi}, \quad g(\varphi\vec{X}, \varphi\vec{Y}) = g(\vec{X}, \vec{Y}) - \eta(\vec{X})\eta(\vec{Y})$$

for all $\vec{X}, \vec{Y} \in \Xi(X)$. It is easy to check that the tensor $\Omega(\vec{X}, \vec{Y}) = g(\vec{X}, \varphi\vec{Y})$ is skew-symmetric. It is called the fundamental tensor of the structure. A manifold with a fixed almost contact

metric structure is called an almost contact metric manifold. If $\Omega = d\eta$ holds, then the almost contact metric structure is called a contact metric structure. An almost contact metric structure is called normal if

$$N_\varphi + 2d\eta \otimes \vec{\xi} = 0,$$

where N_φ is the Nijenhuis torsion defined for the tensor φ. A normal contact metric structure is called a Sasakian structure. A manifold with a given Sasakian structure is called a Sasakian manifold. Let D be the smooth distribution of codimension 1 defined by the form η, and $D^\perp = \operatorname{span}(\vec{\xi})$ be the closing of D. In what follows we assume that the restriction of the 2-form $\omega = d\eta$ to the distribution D is non-degenerate. In this case the vector $\vec{\xi}$ is uniquely defined by the condition

$$\eta(\vec{\xi}) = 1, \quad \ker\omega = \operatorname{span}(\vec{\xi}),$$

and it is called the Reeb vector field. The smooth distribution D we call sometimes a nonholonomic manifold.

For investigation of the intrinsic geometry of a nonholonomic manifold, and generally for the study of almost contact metric structures, it is suitable to use coordinate systems satisfying certain additional conditions. We say that a coordinate map $K(x^\alpha)$ $(\alpha, \beta, \gamma = 1, ..., n)$ $(a, b, c, e = 1, ..., n-1)$ on a manifold X is adapted to the nonholonomic manifold D if

$$D^\perp = \operatorname{span}\left(\frac{\partial}{\partial x^n}\right)$$

holds. It is easy to show that any two adapted coordinate map are related by a transformation of the form

$$x^a = x^a(x^{\tilde{a}}), \quad x^n = x^n(x^{\tilde{a}}, x^{\tilde{n}}).$$

Such coordinate systems are called by Wagner in [10] gradient coordinate systems. Adapted coordinates have their applications in the foliation theory, see e.g. [7].

Let $P : TX \to D$ be the projection map defined by the decomposition $TX = D \oplus D^\perp$ and let $K(x^\alpha)$ be an adapted coordinate map. Vector fields

$$P(\partial_a) = \vec{e}_a = \partial_a - \Gamma^n_a \partial_n$$

are linearly independent, and linearly generate the system D over the domain of the definition of the coordinate map:

$$D = \operatorname{span}(\vec{e}_a).$$

Thus we have on X the nonholonomic field of bases $(\vec{e}_a, \partial_n)$ and the corresponding field of cobases

$$(dx^a, \theta^n = dx^n + \Gamma^n_a dx^a).$$

It can be checked directly that

$$[\vec{e}_a, \vec{e}_b] = M^n_{ab}\partial_n,$$

where the components M^n_{ab} form the so-called tensor of nonholonomicity [10]. Under assumption that for all adapted coordinate systems it holds $\vec{\xi} = \partial_n$, the following equality takes place

$$[\vec{e}_a, \vec{e}_b] = 2\omega_{ba}\partial_n,$$

where $\omega = d\eta$. In what follows we consider exceptionally adapted coordinate systems that satisfy the condition $\vec{\xi} = \partial_n$. We say also that the basis

$$\vec{e}_a = \partial_a - \Gamma^n_a \partial_n$$

is adapted, as the basis defined by an adapted coordinate map. Under the transformation of the adapted coordinate systems, the vectors of the adapted bases transform in the following way: $\vec{e}_a = \frac{\partial x^{\tilde{a}}}{\partial x^a}\vec{e}_{\tilde{a}}$.

We call a tensor field defined on an almost contact metric manifold admissible (to the distribution D) if it vanishes whenever its vectorial argument belongs to the closing $D^\perp$ and its covectorian argument is proportional to the form η. The coordinate form of an admissible tensor field of type (p,q) in an adapted coordinate map looks like

$$t = t^{a_1,\dots,a_p}_{b_1,\dots,b_q} \vec{e}_{a_1} \otimes \dots \otimes \vec{e}_{a_p} \otimes dx^{b_1} \otimes \dots \otimes dx^{b_q}.$$

In particular, an admissible vector field is a vector field tangent to the distribution D, and an admissible 1-form is a 1-form zero on the closing $D^\perp$. It is clear that any tensor structure defined on the manifold X defines on it a unique admissible tensor structure of the same type. From the definition of an almost contact structure it follows that the field of endomorphisms φ is an admissible tensor field of type $(1,1)$. The field of endomorphisms φ we call an admissible almost complex structure. The 2-form $\omega = d\eta$ is also an admissible tensor field. In the geometry of the fibered spaces an admissible tensor field is called semi basic.

Theorem 1. *The derivatives $\partial_n t$ of the components of an admissible tensor field t in an adapted coordinate system are components of an admissible tensor field of the same type.*

The proof of the theorem follows from the fact that the components of an admissible tensor field under the change of an admissible coordinate system transform in the following way:

$$t^{a_1,\dots,a_p}_{b_1,\dots,b_q} = t^{\tilde{a}_1,\dots,\tilde{a}_p}_{\tilde{b}_1,\dots,\tilde{b}_q} A^{a_1}_{\tilde{a}_1} \cdots A^{\tilde{b}_q}_{b_q},$$

where $A^{a_i}_{\tilde{a}_i} = \frac{\partial x^{a^i}}{\partial x^{\tilde{a}^i}}$.

The invariant character of the above statement is enclosed in the equality

$$L_{\vec{\xi}} t^{a_1,\dots,a_p}_{b_1,\dots,b_q} = \partial_n t^{a_1,\dots,a_p}_{b_1,\dots,b_q},$$

where $L_{\vec{\xi}}$ is the Lie derivative along a vector field $\vec{\xi}$.

We say that an admissible tensor field is integrable if there exists an atlas of adapted coordinate maps such that the components of this tensor in any of these coordinate maps are constant. From Theorem 1 immediately follows that the necessary condition of the integrability of an admissible tensor field t is vanishing of the derivatives $\partial_n t$. We call an admissible tensor structure t quasi-integrable if in the adapted coordinates it holds $\partial_n t = 0$. The form $\omega = d\eta$ is an important example of an integrable admissible structure. The following two theorems show the importance of the just now given definitions.

Theorem 2. *The field of endomorphism φ is integrable if and only if $P(N_\varphi) = 0$ holds.*

Proof. $\Rightarrow$: The expression of the Nijenhuis torsion

$$N_\varphi(\vec{X},\vec{Y}) = [\varphi\vec{X},\varphi\vec{Y}] + \varphi^2[\vec{X},\vec{Y}] - \varphi[\varphi\vec{X},\vec{Y}] - \varphi[\vec{X},\varphi\vec{Y}]$$

of the tensor φ in adapted coordinates has the form:

$$N^e_{ab} = \varphi^c_a \vec{e}_c \varphi^e_b - \varphi^c_b \vec{e}_c \varphi^e_a + \varphi^e_c \vec{e}_b \varphi^c_a - \varphi^e_d \vec{e}_a \varphi^d_b, \tag{1}$$

$$N^n_{ab} = 2\varphi^c_a \varphi^d_b \omega_{dc}, \tag{2}$$

$$N^e_{na} = -\varphi^e_c \partial_n \varphi^c_a, \tag{3}$$

$$N^n_{na} = 0, \tag{4}$$

$$N^a_{nn} = 0. \tag{5}$$

If the structure φ is integrable, then from (1)–(5) it follows that

$$N_\varphi(\vec{e}_a,\vec{e}_b) = \varphi^c_a \varphi^d_b M^n_{cd} \partial_n, \quad N_\varphi(\partial_n,\vec{e}_a) = -(\partial_n \varphi^c_b)\varphi^a_c \vec{e}_a.$$

The last two equalities imply $P(N_\varphi) = 0$.

$\Leftarrow$: Suppose that $P(N_\varphi)=0$. Consider sufficiently small neighborhood U of an arbitrary point of the manifold X. Assume that $U=U_1\times U_2$, $TU=\operatorname{span}(\partial_a)\oplus\operatorname{span}(\partial_n)$. We set the natural denotation $T(U_1)=\operatorname{span}(\partial_a)$. We define over the set U the isomorphism of bundles $\psi: D\to T(U_1)$ by the formula $\psi(\vec{e}_a)=\partial_a$. This endomorphism induces an almost complex structure on the manifold U_1. This complex structure is integrable due to the equality $P(N_\varphi)=0$. Indeed, from (3) it follows that the right hand side part of (1) coincides with the torsion of the almost complex structure induced on the manifold U_1. Choosing an appropriate coordinate system on U_1, and consequently an appropriate adapted coordinate system on the manifold X, we get a coordinate map with respect to that the components of the endomorphism field φ are constant. $\square$

Theorem 3. *An almost contact metric structure is normal if and only if the following conditions hold:*

$$P(N_\varphi)=0,\quad \omega(\varphi\vec{u},\varphi\vec{v})=\omega(\vec{u},\vec{v}).$$

Proof. Using the coordinate form (1)–(5), we see that the condition $N_\varphi+2d\eta\otimes\vec{\xi}=0$ is equivalent to the following system of equations:

$$\begin{aligned}\varphi^c_a\vec{e}_c\varphi^e_b-\varphi^d_b\vec{e}_c\varphi^e_a+\varphi^e_c\vec{e}_b\varphi^c_a-\varphi^e_d\vec{e}_a\varphi^d_b&=0,\\ -\varphi^e_c\partial_n\varphi^c_a&=0,\\ 2\varphi^c_a\varphi^d_b\omega_{dc}&=2\omega_{ba}.\end{aligned}\tag{6}$$

This proves the theorem. $\square$

The next statements shows the advisability of the notions like an adapted coordinate system and an integrable tensor field.

Theorem 4. *A contact metric structure is normal if and only if the field of endomorphisms φ is integrable.*

Proof. This statement follows from the fact that for a contact metric structure the condition $N_\varphi+2d\eta\otimes\vec{\xi}=0$ is equivalent to the equality $P(N_\varphi)=0$, since the condition (6), written in the coordinate-free form $\omega(\varphi\vec{u},\varphi\vec{v})=\omega(\vec{u},\vec{v})$ holds automatically due to the definition of the contact metric structure. $\square$

Theorem 4 confirms the importance of the introducing of the new types of the almost contact metric spaces. Namely, we call an almost contact metric space *a Hermitian almost contact metric space* if the condition $P(N_\varphi)=0$ holds.

An intrinsic linear connection on a nonholonomic manifold D is defined in [10] as a map

$$\nabla:\Gamma D\times\Gamma D\to\Gamma D$$

that satisfy the following conditions:

$$\begin{aligned}&1)\quad \nabla_{f_1\vec{u}_1+f_2\vec{u}_2}=f_1\nabla_{\vec{u}_1}+f_2\nabla_{\vec{u}_2};\\ &2)\quad \nabla_{\vec{u}}f\vec{v}=f\nabla_{\vec{u}}\vec{v}+(\vec{u}f)\vec{v},\end{aligned}$$

where ΓD is the module of admissible vector fields. The Christoffel symbols are defined by the relation

$$\nabla_{\vec{e}_a}\vec{e}_b=\Gamma^c_{ab}\vec{e}_c.$$

The torsion S of the intrinsic linear connection is defined by the formula

$$S(\vec{X},\vec{Y})=\nabla_{\vec{X}}\vec{Y}-\nabla_{\vec{Y}}\vec{X}-P[\vec{X},\vec{Y}].$$

Thus with respect to an adapted coordinate system it holds

$$S^c_{ab}=\Gamma^c_{ab}-\Gamma^c_{ba}.$$

In the same way as a linear connection on a smooth manifold, an intrinsic connection can be defined by giving a horizontal distribution over a total space of some vector bundle. The role of such bundle plays the distribution D.

In order to define a connection over the distribution D, it is necessary first to introduce a structure of a smooth manifold on D. This structure is defined in the following way. To each adapted coordinate map $K(x^\alpha)$ on the manifold X we put in correspondence the coordinate map $\tilde{K}(x^\alpha, x^{n+\alpha})$ on the manifold D, where $x^{n+\alpha}$ are the coordinates of an admissible vector with respect to the basis $\vec{e}_a = \partial_a - \Gamma^n_a \partial_n$.

The notion of *a connection over a distribution* introduced in [12, 13], was applied later to nonholonomic manifolds with admissible Finsler metrics in [14, 15]. One says that over a distribution D a connection is given if the distribution $\tilde{D} = \pi_*^{-1}(D)$, where $\pi : D \to X$ is the natural projection, can be decomposed into a direct some of the form

$$\tilde{D} = HD \oplus VD,$$

where VD is the vertical distribution on the total space D. Thus the assignment of a connection over a distribution is equivalent to the assignment of the object $G^a_b(X^a, X^{n+a})$ such that

$$HD = \operatorname{span}(\vec{\epsilon}_a),$$

where $\vec{\epsilon}_a = \partial_a - \Gamma^n_a \partial_n - G^b_a \partial_{n+b}$.

It can be checked in the usual way that that the connection over the distribution D coincides with the linear connection in the nonholonomic manifold D if it holds

$$G^a_b(x^a, x^{n+a}) = \Gamma^a_{bc}(x^a) x^{n+c}.$$

In [15] the notion of the prolonged connection was introduced. The prolonged connection can be obtained from an intrinsic connection by the equality

$$TD = \tilde{HD} \oplus VD,$$

where $HD \subset \tilde{HD}$. Essentially, the prolonged connection is a connection in a vector bundle.

An important example of a manifold with an admissible tensor structure and a compatible with it intrinsic connection considered V.V. Wagner in [10]. In this paper, in a nonholonomic manifold an intrinsic metric is introduced using an admissible tensor field g that satisfies the usual properties of the metric tensor in a Riemannian space.

Similarly to the holonomic case, a metric on a nonholonomic manifold defines there an intrinsic linear symmetric connection. The corresponding Christoffel symbols can be derived from the system of equations

$$\nabla_c g_{ab} = \vec{e}_c g_{ab} - \Gamma^d_{ca} g_{db} - \Gamma^d_{cb} g_{ad}.$$

Let φ be an admissible almost complex structure. We will use the following statement.

Theorem 5. *Each nonholonomic manifold with an almost complex structure φ and an intrinsic torsion-free linear connection ∇ admits an intrinsic linear connection $\tilde{\nabla}$ compatibel with the structure φ and having the torsion S such that*

$$S(\vec{u}, \vec{v}) = \frac{1}{4} P N_\varphi(\vec{u}, \vec{v}),$$

where $\vec{u}, \vec{v} \in \Gamma(D)$.

The proof of this theorem is based on the ideas of the proof of Theorem 3.4 from [16, Ch. 9]. One defines the following connection $\tilde{\nabla}$:

$$\tilde{\nabla}_{\vec{u}} \vec{v} = \nabla_{\vec{u}} \vec{v} - Q(\vec{u}, \vec{v}), \tag{7}$$

where Q can be constructed in a special way using ∇ and φ [16, P. 137].

2. Interior characteristics of almost contact metric spaces

Now we introduce the notion of an almost contact metric structure in a new sense. Namely, we will say that a manifold of almost contact metric structure in the new sense is given if on a manifold X with a given contact form η, in addition a pair of admissible tensor structures (φ, g) such that

$$\varphi^2\vec{u} = -\vec{u}, \quad g(\varphi\vec{u}, \varphi\vec{v}) = g(\vec{u}, \vec{v})$$

is given.

Theorem 6. *The notion of a manifold of almost contact metric structure in the new sense is equivalent to the notion of a manifold of almost contact metric structure in the old sense.*

Proof. Suppose that on the manifold X an almost contact metric structure in the new sense is given. We introduce on the manifold X the Riemannian metric $\tilde{g}$ by the equality

$$\tilde{g}(\vec{u}, \vec{v}) = g(\vec{u}, \vec{v}),$$

where $\vec{u}, \vec{v} \in \Gamma D$, $\tilde{g}(\vec{u}, \vec{\xi}) = 0$ and $\tilde{g}(\vec{\xi}, \vec{\xi}) = 1$. The necessary conditions on field of endomorphisms φ and the 1-form $\tilde{g}$ can be checked directly.

We say that a Sasakian manifold in the new sense is given if on the manifold X with a given contact metric structure, in addition the equality $P(N) = 0$ holds. Theorems 3 and 6 imply the following statement.

Theorem 7. *The notion of a Sasakian manifold in the new sense is equivalent to the notion of a Sasakian manifold in the old sense.*

In this section we use the following notation. As above, admissible almost complex structure and Riemannian metric will be denoted by φ and g, respectively; the symbol ∇ will denote the intrinsic metric connection, and the symbols $\tilde{g}$ and $\tilde{\nabla}$ will denote the metric tensor in the ambient space and its Levi-Civita connection, respectively.

Theorem 8. *A contact metric structure is normal if and only if the structure φ is quasi-integrable and it holds $\nabla\varphi = 0$, where ∇ is an intrinsic metric connection.*

Proof. $\Rightarrow$: From Theorem 3 it follows that $P(N_\varphi) = 0$, and consequently the structure φ is quasi-integrable. Moreover, constructing, using the metric connection, the following connection satisfying

$$S(\vec{u}, \vec{v}) = \frac{1}{4} P N_\varphi(\vec{u}, \vec{v}), \quad \vec{u}, \vec{v} \in \Gamma(D),$$

we get the second statement.

$\Leftarrow$: From (7) it follows that $Q(\vec{u}, \vec{v}) = 0$ and $PN_\varphi(\vec{u}, \vec{v}) = 0$. This and the quasi-integrability of the structure φ imlies the theorem. □

Note that the equality $\nabla\varphi = 0$ is not true if the connection ∇ and the field of endomorphisms φ are considered as the structures defined on the whole manifold, see e.g. [7].

Next suppose that ∇^1 is the extended connection constructed from the intrinsic connection in the following way:

$$\tilde{HD} = HD \oplus \mathrm{span}(\partial_n),$$

here ∂_n is a vector field on the manifold D. The extended connection allows to formulate the next characteristic feature of the integrability of an almost complex structure φ.

Theorem 9. *An almost complex structure φ is integrable if and only if the equality $\nabla^1\varphi = 0$ holds.*

Finally we formulate a statement concerning K-contact manifolds.

Theorem 10. *An almost contact metric structure is a K-contact structure if and only if the metric g is quasi integrable.*

The theorem follows from the following equivalences:

$$L_{\vec{\xi}}\tilde{g} = 0 \Leftrightarrow L_{\vec{\xi}}g = 0 \Leftrightarrow \partial_n g = 0.$$

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